Asset Prices in Consumption and Production Models

Levent Akdeniz† and W. Davis Dechert‡

February 15, 2007

Abstract

In this paper we use a simple model with a single Cobb–Douglas firm and a consumer with a CRRA utility function to show the difference in the equity premia in the production based Brock model and the consumption based Lucas model. With this simple example we show that the equity premium in the production based model exceeds that of the consumption based model by roughly $\beta^{-1} - 1$.

1 Introduction

As was reported by Mehra and Prescott (1985), the equity premium puzzle arises because for no reasonable parameterizations of the Lucas (1978) asset pricing model is the theoretical equity premium as large as the empirically observed one. In a review of the literature on both the equity premium and the risk free rate puzzles, Kocherlakota (1996) states “. . . there is now a vast literature that seeks to resolve these two puzzles.” A number of alternate models have been developed to explain the equity premium puzzle. Among these are models on habit formation, such as Constantinides (1990), Campbell and Cochrane (1999), Boldrin, Christiano, and Fisher (2001), Campbell (2001) and the references cited there. McGrattan and Prescott (June 2001) find that taxes can be a significant determinant of the equity premium. Brock and Turnovsky (1981) show how government policy can have an impact on asset values. In a paper that goes well beyond the equity premium puzzle, Weitzman (September 2004) argues that the modeling should be forward looking, stating “. . . the correct interpretation requires not frequentist objective estimates of the past mean and variance, but rather Bayesian subjective estimates of the future mean and variance.” (Emphasis in the original.)

These models all capture various aspects of human behavior that have an effect on the level of equity premium. However, those that are explicitly designed to explain the equity premium puzzle do not include production. There is a folk theorem that there is an equivalence between production and endowment models, but as expected, the set of objects that are equivalent to each other are not explicitly spelled out. Mehra and Prescott (1985) write, “With our structure, the process on the endowment is exogenous.
and there is neither capital accumulation nor production. Modifying the technology to admit these opportunities cannot overturn our conclusion, because expanding the set of technologies in this way does not increase the set of joint equilibrium processes on consumption and asset prices.”

Geweke (June 1999) raises the point that “The benefits of an analytically rigorous economic theory will be realized only when harnessed to the same high standards for measurement.” He found that “The posterior distribution for the mean of the risk free rate and the equity premium supports values consistent with . . . dynamic stochastic general equilibrium models designed to address this question.” Campbell (2001) comments that “Models with production also help one to move away from the common assumption that stock market dividends equal consumption . . .” and he goes on to conclude that “. . . it will ultimately be more satisfactory to derive both dividends and consumption within a general equilibrium model.” This is, of course, what Brock does in his asset pricing model.

Akdeniz and Dechert (2006) show that there are parameterizations of the Brock (1982) that give much higher equity premia than the consumption based model. In referring to Brock’s asset pricing model, Black (1995) remarked, “If we add non-separable utility, adjustment costs for moving capital from one sector to another, human capital, and a few other features, we will have a model of the kind I favor.” This kind of model would indeed include many useful components that would make the results more realistic. Nevertheless, as was shown by Akdeniz and Dechert, including production alone substantially increases the equity premium. Jermann (1998) shows that in a model with habit formation and production (he uses a capital adjustment cost model) the equity premium can reach 6.16% when the relative risk aversion parameter is 5. This is higher than for models with only habit persistence as the determinant of the equity premium. Dunbar (2006) uses a production model with idiosyncratic shocks that also has a high equity premium. Angeletos and Calvet (2006) use a model with idiosyncratic shocks and incomplete markets to show that values of the equity premium around 4.5% are possible. In all of this research, the production process is an essential element in achieving these results.

In this paper we perform the following thought experiment: use the Brock (1982) model to generate the sequences of consumption, investment and output. Then price the consumption sequence as in Lucas (1978) and compare the equity premia from both models. As we show in section 4 there is a direct analytical link between these two prices, with which we can derive a formula for the equity premia difference. In section 5 we use the standard model with a logarithmic utility function and Cobb–Douglas production function to show that the equity premium in the Brock model is strictly larger than it is in the Lucas model.

2 Consumption Based Asset Prices

In the Lucas (1978) model a consumption stream \( \{c_t\} \) is priced using the first order condition of dynamic optimization,

\[
u'(c_t) \tilde{P}_t = \beta E_t \left[ u'(c_{t+1}) (\tilde{P}_{t+1} + c_{t+1}) \right]
\]

(2.1)

It is notationally useful to define the discounted intertemporal marginal rate of substi-
tution by
\[ \Gamma_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)} \] (2.2)

and to write the pricing equation (2.1) as
\[ \tilde{P}_t = E_t \left[ \Gamma_{t+1} \left( \tilde{P}_{t+1} + c_{t+1} \right) \right] \] (2.3)

Any asset that is in zero net supply can similarly be priced. A one period bond that pays one unit of the consumption good at date \( t + 1 \) will have a price at date \( t \) of
\[ B_t = E_t [\Gamma_{t+1}] \] (2.4)

With equations (2.3) and (2.4) we can compute the consumption based equity premium,
\[ E_t \left[ \frac{\tilde{P}_{t+1}}{\tilde{P}_t} \right] - \frac{1}{B_t} \] (2.5)

3 The One Firm Model

Consider the Brock and Mirman (1972) growth model:

\[
\max_{\{x_t\}} E \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right] 
\]

subject to:
\[ c_t + x_t = y_t \]
\[ y_{t+1} = f(x_t, \xi_{t+1}) \]

where \( c_t \) and \( x_t \) are the consumption and investment at time \( t \), and \( \xi_{t+1} \) is the production shock that affects output, \( y_{t+1} \), at date \( t + 1 \).

Following the method of Brock (1982), we can also use this growth model as an asset pricing model. The consumer’s budget constraint is
\[ c_t + x_t + P_t S_t \leq (\pi_t + P_t) S_{t-1} + r_t x_{t-1} \]

where

- \( P_t \) Share price at date \( t \)
- \( S_t \) Number of shares held at date \( t \)
- \( r_t \) Rental on capital received at date \( t \)
- \( \pi_t \) Profits paid as dividends at date \( t \)
The consumer treats profits (in this model all profits are paid out as dividends) and rents, \( \pi_t \) and \( r_t \), as exogenous to his/her consumption, rental and share purchase decisions, \( c_t \), \( x_t \) and \( S_t \). In a rational expectations equilibrium the following hold:

\[
\begin{align*}
\pi_t &= f(x_{t-1}, \xi_t) - f'(x_{t-1}, \xi_t)x_{t-1} \\
r_t &= f'(x_{t-1}, \xi_t) \\
S_t &= 1 \\
c_t + x_t &= y_t
\end{align*}
\]

The first order equations with respect to \( x_t \) and \( S_t \) for this model are

\[
\begin{align*}
1 &= E_t[\Gamma_{t+1}r_{t+1}] \quad (3.1) \\
P_t &= E_t[\Gamma_{t+1}(P_{t+1} + \pi_{t+1})] \quad (3.2)
\end{align*}
\]

and the equity premium is

\[
E_t \left[ \frac{P_{t+1} + \pi_{t+1}}{P_t} \right] - \frac{1}{B_t} \quad (3.3)
\]

4 The Comparison of Equity Premia

The difference in the equity premium from equations (2.5) and (3.3) is

\[
E_t \left[ \frac{P_{t+1} + \pi_{t+1}}{P_t} - \frac{\hat{P}_{t+1}}{P_t} \right] \quad (4.1)
\]

In order to complete the analysis to see if there is, in fact, a difference in the equity premia in the two models, we need to solve for the relationship between \( \hat{P}_t \) and \( P_t \). Let the optimal solution in a Brock production model have \( \{c_t\} \) as the consumption sequence. Price this sequence with the Lucas model. Its price sequence will be \( \hat{P}_t \). Let \( \{P_t\} \) be the price sequence from the Brock model. We can get a direct comparison of these two price sequences by expanding equation (3.2) and using equation (3.1) and the REE conditions:

\[
P_t = E_t[\Gamma_{t+1}(P_{t+1} + \pi_{t+1})] \\
= E_t[\Gamma_{t+1}(P_{t+1} + y_{t+1} - r_{t+1})] \\
= E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})] - E_t[\Gamma_{t+1}r_{t+1}x_t] \\
= E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})] - x_t
\]

Rewrite this last equation as:

\[
(P_t + x_t) = E_t[\Gamma_{t+1}(P_{t+1} + x_{t+1} + c_{t+1})] \quad (4.2)
\]

Now subtract equation (4.2) from equation (2.3):

4
\[
\left( \tilde{P}_t - P_t - x_t \right) = E_t \left[ \Gamma_{t+1} \left( \tilde{P}_{t+1} - P_{t+1} - x_{t+1} \right) \right]
\]

By the transversality condition, the terms in parentheses are zero, so
\[
\tilde{P}_t = P_t + x_t
\]  
(4.3)

This is the relationship between the asset prices in the two models.

The difference in the returns on these two types of assets is
\[
E_t\left[ \frac{P_{t+1} + \pi_{t+1}}{P_t} - \tilde{P}_{t+1} \right] = E_t\left[ \frac{P_{t+1} + \pi_{t+1}}{P_t} - \frac{P_{t+1} + x_{t+1}}{P_t + x_t} \right]
\]  
(4.4)

We can use this equation to measure the difference in the equity premia in the two models.

### 5 An Example

Consider the well analyzed case of a logarithmic utility function with a Cobb-Douglas production function:

\[
\begin{align*}
  u(c) &= \ln(c) \\
  f(x, \xi) &= \xi x^\alpha
\end{align*}
\]

where \(0 < \alpha < 1\) and \(\xi > 0\). The solution (which is well known) can be found in Brock (1982):

\[
\begin{align*}
  x_t &= \alpha \beta y_t \\
  \pi_t &= (1 - \alpha) y_t \\
  P_t &= \frac{\beta(1 - \alpha)}{1 - \beta} y_t
\end{align*}
\]

With these equations we can solve for the consumption based asset price:

\[
\tilde{P}_t = P_t + x_t = \frac{\beta(1 - \alpha \beta)}{1 - \beta} y_t
\]

Next, compute the difference in the asset returns from equation (4.4):
\[ E_t \left[ \frac{P_{t+1} + \pi_{t+1} - \tilde{P}_{t+1}}{P_t} \right] = E_t \left[ \frac{P_{t+1} + \pi_{t+1} - \tilde{P}_{t+1}}{P_t} \right] \\
= E_t \left[ \frac{y_{t+1}}{y_t} + \frac{1 - \beta}{\beta} \frac{y_{t+1}}{y_t} - \frac{y_{t+1}}{y_t} \right] \\
= (\beta^{-1} - 1) E_t \left[ \frac{y_{t+1}}{y_t} \right] \quad (5.1) \]

Notice that this difference is \textit{positive} wp1. Therefore, the equity premia in the production based asset model is \textit{strictly larger} than it is in the consumption based model. In this example we can compute the conditional expected equity premia difference:

\[ (\beta^{-1} - 1) E_t \left[ \frac{y_{t+1}}{y_t} \right] = (\beta^{-1} - 1) \left( \frac{\alpha \beta^\alpha \bar{\xi}}{y_t^{1-\alpha}} \right) \]

where \( \bar{\xi} = E [\xi_t] \). This difference is larger in the trough of a business cycle than it is at the peak.

6 Numerical Results

In order to calculate the value of the mean difference in the equity premium in the two models over the business cycle, we need to compute the expected value of equation (5.1):

\[ (\beta^{-1} - 1) E \left[ \frac{y_{t+1}}{y_t} \right] \quad (6.2) \]

We simulated the model for values of \( \alpha = 0.4, \beta = 0.97 \) and for the CRRA parameter in the range of 1 – 4. The simulations were for 100,000 time periods and the value of \( E [y_{t+1}/y_t] \) was in the range of 1.002 – 1.003. Thus, for this example the mean value of the \textit{difference} in the equity premium in the two models is

\[ (\beta^{-1} - 1) E \left[ \frac{y_{t+1}}{y_t} \right] = 0.031 \]

7 Conclusion

This example should lay to rest the notion that production \textit{in and of itself} does not add anything to the analysis of asset prices. In this example, there is only one firm with 100% depreciation. In Akdeniz and Dechert (2006) it was shown that for a model with 4 firms, with partial depreciation and for which the technology parameters are state dependent the equity premium can easily be twice the value of \( \beta^{-1} - 1 \). This is due, in part, to the fact that such a model also can include idiosyncratic risk which is also a determinant of the equity premium.
A point that should not be lost on the reader, is that any research on asset prices that uses a representative consumer model can include a comparison of the consumption based prices with the asset prices in the model. As our equations (2.2) — (2.3) show, one only needs only the consumption stream generated by the model to compute the Lucas asset prices. Comparisons, such as equation (3.3) with (2.5) can be readily made and reported.

References


